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GENERALIZED HYPERGEOMETRIC FUNCTIONS AND THEIR APPLICATIONS IN SOLVING SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract: The generalized hypergeometric function is one of the most important classes of special functions in mathematical analysis and has extensive applications in solving differential equations arising in mathematics, physics, and engineering. In this paper, the fundamental properties of generalized hypergeometric functions are studied and their role in obtaining analytical solutions of second-order linear differential equations is investigated. Various special functions such as Gauss hypergeometric functions, confluent hypergeometric functions, Bessel functions, Legendre polynomials, Laguerre polynomials, and Hermite polynomials are represented in terms of generalized hypergeometric functions. The corresponding second-order differential equations are transformed into hypergeometric form, and their exact solutions are obtained through generalized hypergeometric series representations. The results demonstrate that generalized hypergeometric functions provide a unified and effective framework for studying a wide range of differential equations and establish important connections among different classes of special functions.

Keywords: Differential Equations, Second-Order Linear, Generalized Hypergeometric Functions, Applications, Mathematical Functions.

1. INTRODUCTION

Special functions play a significant role in modern mathematics and its applications. They arise naturally in the solutions of differential equations that model various physical phenomena in quantum mechanics, fluid dynamics, heat conduction, electromagnetic theory, and engineering sciences. Many problems encountered in these fields cannot be solved by elementary functions alone, leading to the development and study of special functions as powerful analytical tools.

Among the various classes of special functions, hypergeometric functions occupy a central position due to their rich mathematical structure and wide applicability. The generalized hypergeometric function extends the classical hypergeometric series and provides a unified framework for representing numerous special functions, including Bessel functions, Legendre functions, Hermite polynomials, Laguerre polynomials, and Jacobi polynomials. Because of this unifying nature, generalized hypergeometric functions have become an important area of research in mathematical analysis and applied mathematics.

Second-order linear differential equations frequently arise in mathematical physics and engineering applications. Many classical differential equations can be transformed into hypergeometric form, allowing their solutions to be expressed in terms of generalized hypergeometric functions. This approach not only provides exact analytical solutions but also establishes relationships among different families of special functions. The present work is concerned with the study of generalized hypergeometric functions and their applications in solving second-order linear differential equations. The generalized hypergeometric function is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!}, \qquad (1.1.1) $$

Where

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \qquad (a)_0 = 1, \qquad (1.1.2) $$

and $\Gamma(\cdot)$ denotes the Gamma function.

Introducing the differential operator

$$ \theta = z\frac{d}{dz}, \qquad (1.1.3) $$

the generalized hypergeometric function satisfies the differential equation

$$ \left[\theta(\theta+b_1-1)(\theta+b_2-1)\cdots(\theta+b_q-1) - z(\theta+a_1)(\theta+a_2)\cdots(\theta+a_p)\right] y(z) = 0 \qquad (1.1.4) $$

which forms the basis for studying many special functions.

A general second-order linear differential equation is given by

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0, \qquad (1.1.5)$$

where P(x), Q(x), and R(x) are continuous functions.

One of the most important examples is the Gauss hypergeometric differential equation

$$z(1-z)\frac{d^2y}{dz^2} + \left[c - (a+b+1)z\right]\frac{dy}{dz} - ab\,y = 0, \qquad (1.1.6)$$

whose solution is

$$y(z) = {}_2F_1(a,b;c;z). \qquad (1.1.7)$$

Similarly, Bessel's differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \qquad (1.1.8)$$

has the hypergeometric representation

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-;\nu+1;-\frac{x^2}{4}\right). \qquad (1.1.9)$$

between generalized hypergeometric functions and second-order linear differential equations, making them powerful tools in mathematical analysis and its applications.

1.1 Importance of Generalized Hypergeometric Functions

The generalized hypergeometric function is one of the most important special functions in mathematical analysis. It arises naturally in many branches of pure and applied mathematics and serves as a powerful tool for obtaining analytical solutions of differential equations.

1.2 Representation of Special Functions

Generalized hypergeometric functions can represent numerous special functions as particular cases. Some important examples include:

1.2.1 Bessel Functions

Bessel functions can be expressed in terms of generalized hypergeometric functions and are widely used in problems involving cylindrical symmetry.

1.2.2 Legendre Functions

Legendre functions arise in solving Laplace's equation in spherical coordinates and admit hypergeometric representations.

1.2.3 Laguerre and Hermite Polynomials

Laguerre and Hermite polynomials can be represented as hypergeometric functions and play an important role in quantum mechanics and mathematical physics.

1.3 Solution of Differential Equations

A remarkable property of generalized hypergeometric functions is that they satisfy linear differential equations. Many classical second-order differential equations can be transformed into hypergeometric form, making it possible to obtain exact analytical solutions.

1.4 Applications in Mathematical Physics

Generalized hypergeometric functions occur in a variety of applications including quantum mechanics, wave propagation, heat conduction, electromagnetic theory, and potential theory. Their ability to represent solutions of differential equations makes them indispensable in mathematical modeling.

1.5 Objective of the Study

The main objective of this paper is to study generalized hypergeometric functions and investigate their applications in solving second-order linear differential equations. Particular emphasis is placed on expressing classical differential equations in hypergeometric form and obtaining their exact solutions through generalized hypergeometric series.

2. PROPERTIES OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

The derivative formula is given by

$$\frac{d}{dz}\left[{}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)\right]=\frac{a_1a_2\cdots a_p}{b_1b_2\cdots b_q}\,{}_pF_q\left(\begin{matrix}a_1+1,\ldots,a_p+1\\ b_1+1,\ldots,b_q+1\end{matrix};z\right),\qquad(2.1.1)$$

The recurrence relation for the Pochhammer symbol is

$$(a)_{n+1}=(a+n)(a)_n,\qquad(2.1.2)$$

Also,

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)},\qquad(2.1.3)$$

The Euler transformation formula is

$${}_2F_1(a,b;c;z)=(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z),\qquad(2.1.4)$$

The Pfaff transformation formula is

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),\qquad(2.1.5)$$

The confluent hypergeometric function is represented by

$${}_1F_1(a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!},\qquad(2.1.6)$$

It satisfies

$$zy''+(c-z)y'-ay=0,\qquad(2.1.7)$$

The Bessel function representation is

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right), \qquad (2.1.8)$$

The Legendre polynomial can be expressed as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1(-n; \alpha+1; x), \qquad (2.1.9)$$

The Laguerre polynomial is

$$H_n(x) = (2x)^n {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; -; -\frac{1}{x^2}\right), \qquad (2.1.10)$$

3. APPLICATIONS TO SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

The generalized hypergeometric function provides a unified approach for solving a variety of second-order linear differential equations. Several classical equations of mathematical physics can be expressed in hypergeometric form.

The Gauss hypergeometric differential equation is

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - ab\,y = 0, \qquad (3.1.1)$$

whose solution is

$$y(z) = C_1\, {}_2F_1(a, b; c; z) + C_2\, z^{1-c} {}_2F_1(a-c+1, b-c+1; 2-c; z), \qquad (3.1.2)$$

The confluent hypergeometric differential equation is

$$z\frac{d^2y}{dz^2} + (c - z)\frac{dy}{dz} - ay = 0, \qquad (3.1.3)$$

with solution

$$y(z) = {}_1F_1(a; c; z), \qquad (3.1.4)$$

Consider Legendre's differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \qquad (3.1.5)$$

whose solution may be written as

$$P_n(x) = {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right), \qquad (3.1.6)$$

Similarly, Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-\nu^2)y = 0, \qquad (3.1.7)$$

and its solution is

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-; \nu+1; -\frac{x^2}{4}\right), \qquad (3.1.8)$$

The modified Bessel equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)y = 0, \qquad (3.1.9)$$

with solution

$$I_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\left(-; \nu+1; \frac{x^2}{4}\right), \qquad (3.1.10)$$

The Laguerre differential equation is

$$x\frac{d^2y}{dx^2} + (\alpha+1-x)\frac{dy}{dx} + ny = 0, \qquad (3.1.11)$$

whose solution is

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1(-n; \alpha+1; x), \qquad (3.1.12)$$

The Hermite differential equation is

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, \qquad (3.1.13)$$

and its solution is

$$H_n(x) = (2x)^n \, {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; -; -\frac{1}{x^2}\right), \qquad (3.1.14)$$

These examples demonstrate that generalized hypergeometric functions provide a powerful framework for expressing exact solutions of second-order linear differential equations arising in mathematical physics and applied mathematics.

4. RESULTS

The preceding sections show that several important second-order linear differential equations can be represented in terms of generalized hypergeometric functions. The Gauss hypergeometric equation admits the solution

$$I_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)} \, {}_0F_1\left(-; \nu+1; \frac{x^2}{4}\right), \qquad (4.1.1)$$

while the confluent hypergeometric equation yields

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2+\nu^2)y = 0, \qquad (4.1.2)$$

The Legendre polynomial is represented as

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_1F_1(-n; \alpha+1; x), \qquad (4.1.3)$$

and the Bessel function takes the form

$$x\frac{d^2y}{dx^2} + (\alpha+1-x)\frac{dy}{dx} + ny = 0, \qquad (4.1.4)$$

demonstrating the close relationship between special functions and generalized hypergeometric functions.

The Laguerre polynomial is given by

$$H_n(x) = (2x)^n \, {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; -; -\frac{1}{x^2}\right), \qquad (4.1.5)$$

whereas the Hermite polynomial may be expressed as

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0, \qquad (4.1.6)$$

showing that many classical orthogonal polynomials arise as special cases of generalized hypergeometric functions.

The generalized hypergeometric differential equation

$$C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{n!} \, {}_2F_1\left(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right), \qquad (4.1.7)$$

$$(1-x^2)\frac{d^2y}{dx^2} - (2\lambda+1)x\frac{dy}{dx} + n(n+2\lambda)y = 0, \qquad (4.1.8)$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \, {}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right), \qquad (4.1.9)$$

$$(1-x^2)\frac{d^2y}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{dy}{dx} + n(n+\alpha+\beta+1)y = 0, \qquad (4.1.10)$$

provides a unified framework for studying differential equations and their solutions.

These results confirm that generalized hypergeometric functions serve as powerful analytical tools for obtaining exact solutions of a wide variety of second-order linear differential equations occurring in mathematical physics and applied mathematics.

The study demonstrates that generalized hypergeometric functions provide a unified mathematical framework for solving a broad class of second-order linear differential equations. It is found that several important special functions, including Gauss hypergeometric functions, confluent hypergeometric

functions, Bessel functions, Legendre polynomials, Laguerre polynomials, Hermite polynomials, Gegenbauer polynomials, and Jacobi polynomials, can be represented as particular cases of generalized hypergeometric functions. The investigation further shows that many classical differential equations arising in mathematical physics can be transformed into hypergeometric form, enabling the derivation of exact analytical solutions. The results confirm that generalized hypergeometric functions simplify the solution process, establish connections among different families of special functions, and serve as an effective tool for studying differential equations encountered in mathematics, physics, and engineering applications.

5. DISCUSSION

The present study demonstrates that generalized hypergeometric functions provide a powerful and unified framework for solving a wide range of second-order linear differential equations. The analysis shows that many classical differential equations, including the Gauss hypergeometric, confluent hypergeometric, Bessel, Legendre, Laguerre, Hermite, Gegenbauer, and Jacobi equations, can be expressed in hypergeometric form and solved through generalized hypergeometric series representations. The obtained results highlight the close relationship between special functions and differential equations, revealing that numerous special functions arise as particular cases of generalized hypergeometric functions. Furthermore, the study confirms that the hypergeometric approach simplifies the derivation of exact analytical solutions and provides a common mathematical structure for investigating problems in mathematical physics, engineering, and applied mathematics. Thus, generalized hypergeometric functions serve as an effective tool for both theoretical analysis and practical applications involving second-order linear differential equations.

6. CONCLUSION

The present study highlights the significance of generalized hypergeometric functions as powerful analytical tools for solving second-order linear differential equations. It has been shown that many classical equations, including the Gauss hypergeometric, confluent hypergeometric, Bessel, Legendre, Laguerre, and Hermite equations, can be expressed and solved within the generalized hypergeometric framework. The obtained results reveal the unifying nature of generalized hypergeometric functions and their ability to represent a broad family of special functions through a common mathematical structure. Consequently, generalized hypergeometric functions not only simplify the solution process of differential equations but also provide deeper insight into the relationships among various special

functions. These properties make them valuable in both theoretical investigations and practical applications arising in mathematical physics, engineering, and applied mathematics.]

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